

4

AD-A222 375

DTRC-90/012 Temporal Response of Coupled One-Dimensional Dynamic Systems

## David Taylor Research Center

Bethesda, MD 20084-5000

DTRC-90/012 April 1990

Propulsion and Auxiliary Systems Department  
Research & Development Report

### Temporal Response of Coupled One-Dimensional Dynamic Systems

by  
J. Dickey  
G. Maidanik

DTIC  
ELECTE  
JUN 04 1990  
S B D  
*Coc*



Approved for public release; distribution is unlimited.

## MAJOR DTRC TECHNICAL COMPONENTS

CODE 011 DIRECTOR OF TECHNOLOGY, PLANS AND ASSESSMENT

12 SHIP SYSTEMS INTEGRATION DEPARTMENT

14 SHIP ELECTROMAGNETIC SIGNATURES DEPARTMENT

15 SHIP HYDROMECHANICS DEPARTMENT

16 AVIATION DEPARTMENT

17 SHIP STRUCTURES AND PROTECTION DEPARTMENT

18 COMPUTATION, MATHEMATICS & LOGISTICS DEPARTMENT

19 SHIP ACOUSTICS DEPARTMENT

27 PROPULSION AND AUXILIARY SYSTEMS DEPARTMENT

28 SHIP MATERIALS ENGINEERING DEPARTMENT

### DTRC ISSUES THREE TYPES OF REPORTS:

1. **DTRC reports, a formal series**, contain information of permanent technical value. They carry a consecutive numerical identification regardless of their classification or the originating department.
2. **Departmental reports, a semiformal series**, contain information of a preliminary, temporary, or proprietary nature or of limited interest or significance. They carry a departmental alphanumeric identification.
3. **Technical memoranda, an informal series**, contain technical documentation of limited use and interest. They are primarily working papers intended for internal use. They carry an identifying number which indicates their type and the numerical code of the originating department. Any distribution outside DTRC must be approved by the head of the originating department on a case-by-case basis.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

## REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION <b>Unclassified</b>			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION/AVAILABILITY OF REPORT  Approved for public release; distribution is unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE					
4. PERFORMING ORGANIZATION REPORT NUMBER(S)  DTRC-90/012			5. MONITORING ORGANIZATION REPORT NUMBER(S)		
6a. NAME OF PERFORMING ORGANIZATION  David Taylor Research Center		6b. OFFICE SYMBOL (If applicable)  Code 2704		7a. NAME OF MONITORING ORGANIZATION	
6c. ADDRESS (City, State, and ZIP Code)  Annapolis, MD 21402			7b. ADDRESS (City, State, and ZIP Code)		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION  David Taylor Research Center		8b. OFFICE SYMBOL (If applicable)  Code 2704		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER	
8c. ADDRESS (City, State, and ZIP Code)  Annapolis, MD 21402			10. SOURCE OF FUNDING NUMBERS		
			PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.
			WORK UNIT ACCESSION NO.		
11. TITLE (Include Security Classification)  Temporal Response of Coupled One-Dimensional Dynamic Systems					
12. PERSONAL AUTHOR(S)  J. Dickey and G. Maidanik					
13a. TYPE OF REPORT  Research & Development		13b. TIME COVERED  FROM _____ TO _____		14. DATE OF REPORT (YEAR, MONTH, DAY)  1990 April 6	
15. PAGE COUNT  35					
16. SUPPLEMENTARY NOTATION  This paper will also be submitted for publication in the open literature.					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB-GROUP	Transient propagation, Pulse propagation, Multiple dynamic systems, Dynamic systems.		
19. ABSTRACT (Continue on reverse if necessary and identify by block number)  A formalism is presented which describes the response of a complex of coupled one-dimensional dynamic systems to an impulse drive. The formalism is based on an impulse response operator which relates a drive applied to one point in the complex to the response at any point in the complex. The formalism is derived directly in the time domain, and the impulsive drives which can be accommodated must be finite in time and applied at a spatial point. The constituent systems must be one-dimensional and possess a pulse propagation velocity which is not a function of position within the system. Systems interact through junctions which also define their spatial extents. The junctions are characterized by reflection and transmission coefficients which modulate the amplitude of reverberant components and by delays in the reflections and transmissions. Propagation in the systems is characterized by losses. Several simplistic examples are calculated and presented to illustrate the type of information which the formalism can provide.					
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS			21. ABSTRACT SECURITY CLASSIFICATION  Unclassified		
22a. NAME OF RESPONSIBLE INDIVIDUAL  J. Dickey			22b. TELEPHONE (Include Area Code)  (301) 267-2759		22c. OFFICE SYMBOL  Code 2704

DD FORM 1473, JUN 86

Previous editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

0102-LF-014-6602

# CONTENTS

	Page
ABSTRACT .....	1
ADMINISTRATIVE INFORMATION.....	1
INTRODUCTION .....	1
THE DELAY OPERATOR.....	5
THE SINGLE SYSTEM.....	6
RESPONSE OF SINGLE SYSTEM: AN EXAMPLE .....	10
MULTIPLE SYSTEMS .....	13
RESPONSE OF MULTIPLE SYSTEMS: AN EXAMPLE.....	15
FIGURES.....	18
REFERENCES.....	31



<b>Accession For</b>	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
<b>Availability Codes</b>	
Dist	Avail and/or Special
A-1	

## ABSTRACT

A formalism is presented which describes the response of a complex of coupled one-dimensional dynamic systems to an impulse drive. The formalism is based on an impulse response operator which relates a drive applied to one point in the complex to the response at any point in the complex. The formalism is derived directly in the time domain and the impulsive drives which can be accommodated must be finite in time and applied at a spatial point. The constituent systems must be one-dimensional and possess a pulse propagation velocity which is not a function of position within the system. Systems interact through junctions which also define their spatial extents. The junctions are characterized by reflection and transmission coefficients which modulate the amplitude of reverberant components and by delays in the reflections and transmissions. Propagation in the systems is characterized by losses. Several simplistic examples are calculated and presented to illustrate the type of information which the formalism can provide.

## ADMINISTRATIVE INFORMATION

This work was supported by the Propulsion and Auxiliary Systems Department, Code 27, of the David Taylor Research Center.

## INTRODUCTION

This paper introduces a formalism which describes the temporal response of a complex consisting of multiple connected, one-dimensional systems. The formalism is based on a matrix impulse response operator

$$\underline{g}(\underline{x} | \underline{x}', t | t') = (g_{ji}(x_j | x_i', t | t')) \quad (1)$$

which depends solely on the properties of the complex. In particular,  $\underline{g}$  depends on the propagation properties of each of the systems, the boundary conditions for each system, the spatial

coordinate and time of the observation and drive in each system. It is assumed that the impedance operator governing the system is linear so that the response of the complex  $\underline{p}(\underline{x}, t)$ , at some point  $\underline{x}$  and at time  $t$ , can be given by the operation of  $\underline{g}$  on an external drive, (i.e. one which does not interfere with or depend on the response of the complex), in the usual way:

$$\begin{aligned} \underline{p}(\underline{x}, t) &= \int \underline{g}(\underline{x} | \underline{x}', t | t') d\underline{x}' dt' \underline{p}_e(\underline{x}', t') , \\ d\underline{x}' &= dx'_i \delta_{ij} . \end{aligned} \quad (2)$$

The rank of the matrix and vector quantities in equation (2) equals the number of systems in the complex, and individual components of  $\underline{p}$ , say  $p_i(x_i, t)$ , describe the response in the (i)th system at the point  $x_i$  and at time  $t$ . Similarly, each component of the drive, e.g.  $p_{ej}(x_j, t)$  describes the drive in one system, e.g. the (j)th, at a point, e.g.  $x_j$ , and at time  $t$ . Each component of the vector spatial variables  $\underline{x}$  and  $\underline{x}'$  is the spatial positions of observation and drive respectively in each of the systems. The response due to a drive in the (j)th system at the point  $x'_j$  and at time  $t'$  is given by the element  $g_{ij}(x_i | x'_j, t | t')$ , the transfer impulse response operator, which makes the connection and completely accounts for the dynamic interaction of all members of the complex.

The focus here is on the temporal response of the complex, and the development of the appropriate impulse response operator will closely parallel the derivation of the spatial impulse response function [1,2]. This approach tracks a single spectral component in the response as this component interacts with the complex and superposes the results of these interactions at an observation point. One of the useful characteristics which the present formalism shares with this approach is that the impulse response operator, and hence the response, so derived, is obtained as a sum of two direct terms and four reverberant terms. The reverberant terms express the response arriving at the observation point from each direction, and in general, each of these are the superposition of two components which left the source in each direction. This division into components can be shown by writing equation (2) in the expanded form:

$$\begin{aligned}
\underline{p}^\alpha(\underline{x}, t) &= \int \underline{\hat{g}}^\alpha(\underline{x} | \underline{x}', t | t') d\underline{x}' dt' \underline{\hat{p}}_e(\underline{x}', t') , \\
\underline{p}(\underline{x}, t) &= \{p_j(x_j, t)\}, \quad \underline{p}_e(\underline{x}', t') = \{p_{ei}(x'_i, t')\} , \\
\underline{g}(\underline{x} | \underline{x}', t | t') &= (g_{ji}(x_j | x'_i, t | t')) ,
\end{aligned} \tag{3a}$$

where  $\alpha = r$  or  $q$ ,  $r$  and  $q$  indicating opposite ends or directions in a system. Thus for example,  $p_j^r(x_j, t)$  is the response in the  $(j)$ th system which is traveling towards the  $r$  junction. The external drive in each system is composed of two components, one launched in each direction,

$$\underline{\hat{p}}_e = \sum_{\alpha=r,q} \underline{\alpha} \underline{p}_{e\alpha} . \tag{3b}$$

The two components of the impulse response operator in equation (3a) are further composed of components which relate drive components emanating in each direction to the arrival components; specifically,

$$\underline{\hat{g}}^\alpha = (\underline{g}_{\infty}^\alpha + \underline{g}_{\alpha}^\alpha) \underline{\alpha} + \underline{g}_{\beta}^\alpha \underline{\beta} , \tag{3c}$$

where, if  $\alpha = r$  then  $\beta = q$  and vice-versa, and  $\underline{g}_{\infty}^\alpha$  is the direct component of response; i.e. the response which has not interacted with the boundaries. Also, for example, the  $\underline{g}_{\beta}^\alpha$  component of the impulse response function relates the  $\beta$  component of the drive with the  $\alpha$  component of the response.

When formulating the response in this way, it is often assumed that the complex is stationary in time. Under this condition one can write

$$\underline{g}(\underline{x} | \underline{x}', t | t') \rightarrow (2\pi)^{-1/2} \underline{g}(\underline{x} | \underline{x}', t - t') . \tag{4}$$

Equation (2) can then be Fourier transformed in the temporal variable yielding

$$\underline{\tilde{p}}(\underline{x}, \omega) = \int \underline{\tilde{g}}(\underline{x} | \underline{x}', \omega) d\underline{x}' \underline{\tilde{p}}_e(\underline{x}', \omega) \tag{5}$$

where the frequency,  $\omega$ , is the Fourier conjugate of the temporal variable,  $t$ , and  $\tilde{p}(\underline{x}, \omega)$ ,  $\tilde{p}_e(\underline{x}, \omega)$  and  $\tilde{g}(\underline{x} | \underline{x}', \omega) \delta(\omega - \omega')$  are the Fourier transforms of  $p(\underline{x}, t)$ ,  $p_e(\underline{x}, t)$  and  $(2\pi)^{-1/2} \tilde{g}(\underline{x} | \underline{x}', t - t')$  respectively. It is then observed that under these conditions of temporal stationarity, equation (2) can be transformed to an equation which is *algebraic* in the frequency. Without this stationarity, equation (5) must remain *operational* in the frequency variable. In the present development, the impulse response operator will be manipulated in operator form and the complex will *not* be considered to be stationary in time. The Fourier transformation in that variable will not be performed; rather we will consider the nature of  $\tilde{g}$  in the temporal domain with the drive points and the observation points held constant. The impedance operator for the complex is linear; thus the response to a spatially distributed source can be constructed as a summation (integration) in the spatial domain in an analogous way in which the Fourier components are conventionally summed (integrated) to get the response to a drive distributed in time.

Equation (2) will hereafter be written as

$$\underline{p}(\underline{x}, t) = \underline{h}(\underline{x}, \underline{x}', t | t') \underline{p}_e(\underline{x}', t') \quad , \quad (6a)$$

with  $\underline{h}$  being termed the impulse response operator. The major object of this paper is to present an explicit form for  $\underline{h}$  and to explore some of its characteristics in the temporal domain by picking fixed drive and observation points, thus effectively "putting a hold" on its spatial characteristics. A comprehensive formalism which treats the spatial and temporal aspects in a unified manner is in preparation [3].

The temporal impulse response operator for a single system would be the same as equation (5) but the matrix and vector quantities would all be of rank one, i.e. for a single system,

$$p(x, t) = h(x, x', t | t') p_e(x', t') \quad . \quad (6b)$$

Since the response to a temporal pulse is more easily visualized in a single system than in a complex, the formalism will first be developed and demonstrated by calculation for a single system. After this, the extension to multiple systems is straightforward; this will be done and demonstrated in the final two sections. There are several assumptions about the systems which are basic to both the single and multiple system formulation. Specifically, the impedance operator which governs the system is assumed to be linear so that solutions may be superposed, and it is assumed that the systems are one-dimensional; i.e. if the physical system being modeled is spatially two or three dimensional, then the impedance operator for the system must be separable in such a way that the spatial dimension of interest can be treated independent of the others. Also, a system can have only one propagation speed and this must be constant throughout the spatial extent of the system. Separate systems can, of course, possess different speeds, and within a particular system, the speed in one direction may differ from the speed in the other direction. In view of this last assumption, a physical system with more than one mode of propagation, and therefore more than one type of pulse propagation, would be formulated as several systems, one for each mode, which are allowed to interact with each other only at the junctions.

## THE DELAY OPERATOR

The temporal impulse response operator,  $h$  in equation (6), uses the delay associated with the propagation of a pulse between two spatial points,  $x'$  and  $x$ , as the "building block" in an analogous way in which the spatial formalism [1,2] uses spatial propagators. The spatial formalism defined a spatial propagator in terms of the eigenfunctions,  $\phi(x)$ , for a specific response mode of a spatially one-dimensional system with no boundaries. This propagator is of the form: [4]

$$t(x | x') = \phi(x')^{-1} \phi(x) \quad . \quad (7)$$

For purposes of constructing the temporal impulse response operator, we consider the delay experienced by the propagation of a pulse in a system which has the same propagation characteristics as the system of interest but which has no boundaries; i.e. a specific response mode in a spatially one-dimensional system which is infinite in extent. If the propagation speed of this pulse is  $u$ , then the delay in propagating from the point  $x'$  to  $x$  is

$$\tau(x | x') = |x' - x| / u \quad . \quad (8)$$

We define a "delay operator",  $\Delta(\tau)$ , such that

$$\begin{aligned} \Delta(\tau) f(t) &= f(t + \tau) , \\ \Delta(\tau_1) \Delta(\tau_2) &= \Delta(\tau_1 + \tau_2) \quad . \end{aligned} \quad (9)$$

The following sections will use the above delay operator to construct the temporal impulse response operator, first for a single system, and then for a complex of connected systems.

## THE SINGLE SYSTEM

A single dynamic system is depicted in Figure 1. The system is defined to have a single mode of propagation and a velocity  $u$  associated with the propagation of a pulse; this velocity is not a function of the spatial coordinate  $x$ . In general, there may exist different velocities associated with the two different directions of propagation within the system, i.e.  $u^r$  and  $u^q$  with  $u^r \neq u^q$  where the superscript on  $u$  indicates the direction towards which the pulse propagates. [cf. Figure 1.] The velocity  $u^\alpha$ ,  $\alpha = r$  or  $q$ , is complex,  $u^\alpha = u_0^\alpha (1 - i\eta_\alpha)$  where  $\eta_\alpha$  plays the role of a "loss factor" associated with propagation.<sup>1</sup>

---

<sup>1</sup>For a system which is non-dispersive,  $k = \omega/c_p$  where  $c_p$  is a phase velocity independent of  $\omega$ , then  $u = c_p$ , and setting  $u = u_0 (1 - i\eta)^{-1}$  is equivalent to the more conventional practice of incorporating a loss factor in the wavenumber,  $k = k_0 (1 - i\eta)$ . When dispersion is present, the equivalence breaks down and the loss factor associated with the group velocity would not be the same as the one

The end points of the system are determined by the coordinate values  $\{x_q, x_r\}$ , hence the length of the system is  $L = |x_r - x_q|$  and the time for propagating a pulse over the entire system is  $\tau(x_r|x_q) = L/u^r$  in the  $q \rightarrow r$  direction and  $\tau(x_q|x_r) = L/u^q$  in the  $r \rightarrow q$  direction. The end points or boundaries of the system are characterized by reflection coefficients,  $\Lambda_r$  and  $\Lambda_q$ , and by delays experienced by an incident pulse during reflection,  $\tau_r$  and  $\tau_q$ . Both the reflection coefficients and the junction (reflection) delays may be complex.

An impulsive drive, i.e. one finite in time, is applied to the system at  $x'$ . In the temporal coordinate, the impulse is centered about the point  $t'$ . The response of the system may be viewed as a spatially finite wave packet whose "center of gravity" propagates away from  $x'$ , in both directions, with velocities  $u^r$  and  $u^q$ . The point of view adopted for the current development is to pick an observation point  $x$  and derive the response as a function of time; i.e., we seek to derive the operator  $h$  of equation (6).

The impedance operator for the system is assumed to be linear and the response is sought as a superposition of a "direct" response and a "reverberant" response, with a corresponding division of the operator  $h$ ,

$$h(x, x', t | t') = h_d(x | x', t | t') + h_{rev}(x | x', t | t') \quad . \quad (10)$$

The "direct" response is given by using  $h_d$  in equation (7) and is the response which emanates from the drive at  $\{x', t'\}$  and propagates, without boundary interaction, to  $\{x, t\}$ . Using the delay operator (11), the direct response is,

$$\begin{aligned} p(x, t) &= h_d(x, x', t | t') p_e(x', t') \\ &= \Delta[\tau(x|x')] p_{er}(x', t') U(x-x') + \Delta[\tau(x|x')] p_{eq}(x', t') U(x'-x) \quad , \quad (11a) \end{aligned}$$

---

*associated with the wavenumber and would need to be determined based on the dispersion characteristics of the system. For purposes of illustrating the formalism, a simple loss factor,  $\eta$ , is associated with  $u$ .*

where the subscripts r and q on the drive,  $p_{\alpha}(x', t)$ , differentiate between drive components which emanate from the point  $x'$  towards the r and q junctions respectively. In general,  $p_r \neq p_q$ . Also in equation (11a), it is assumed that  $x_r > x_q$  and the step function,

$$U(x' - x) = \begin{cases} 1, & x' > x \\ 0, & x' < x \end{cases} \quad (11b)$$

is used to select the drive component which is operative in the direct response.

The reverberant contributions in equation (10) can be built up by referring to Figure (1) and noting that the response at the point x which has been generated by the r- component of the drive and which has interacted once with the r- junction is

$$p_r^q(x, t) = \Delta[\tau(x|x_r)] \Lambda_r \Delta(\tau_r) \Delta[\tau(x_r|x')] p_{er}(x', t') \quad (12)$$

The superscript, q, on the response indicates that the response is propagating toward the q junction and the subscript, r, indicates that the response was generated by the r- component of the drive. In general, there is an infinite family of such responses representing successive reverberations whose arrival times have been delayed by successive complete "reverberation times" for the system and have been attenuated by a propagation loss and successive reflections at the boundaries. The reverberation time for the system is,

$$\begin{aligned} \tau_{sys} &= \tau_q + \tau_{qr} + \tau_r + \tau_{rq} \quad , \\ \tau_{rq} &= \tau(x_r|x_q) = L/u^r, \quad \tau_{qr} = \tau(x_q|x_r) = L/u^q \quad . \end{aligned} \quad (13a)$$

A reverberation operator is defined, which accounts for the successive reverberation delays and the total propagation loss, and the successive losses at the boundaries:

$$\begin{aligned} \mathcal{D} &= 1 + \Lambda_r \Lambda_q \Delta(\tau_{sys}) + [\Lambda_r \Lambda_q \Delta(\tau_{sys})]^2 + [\dots]^3 + \dots \\ &= \sum_{n=0}^{\infty} [\Lambda_r \Lambda_q \Delta(\tau_{sys})]^n \quad . \end{aligned} \quad (13b)$$

Thus, equation (12) [which represents only the first return of the drive component which leaves the point  $x'$  towards the  $r$ -junction, reflects at this junction, and goes to the point  $x$ ] can be modified to incorporate all the reverberations; i.e.

$$p_r^q(x, t) = \Delta[\tau(x|x_r)] \oslash \Lambda_r \Delta(\tau_r) \Delta[\tau(x_r|x')] p_{er}(x', t') \quad (14a)$$

Equation (14a) is one of the four components in the reverberant response. In general, one can formulate a distinct response for each arrival direction and each of these can again be divided into components which have emanated from the drive in the two directions. The remaining three components of the reverberant response can be obtained in a similar way. They are:

$$p_q^q(x, t) = \Delta[\tau(x|x_r)] \Lambda_r \Delta(\tau_r) \Delta[\tau(x_r|x_q)] \oslash \Lambda_q \Delta(\tau_q) \Delta[\tau(x_q|x')] p_{eq}(x', t') \quad (14b)$$

$$p_r^r(x, t) = \Delta[\tau(x|x_q)] \Lambda_q \Delta(\tau_q) \Delta[\tau(x_q|x_r)] \oslash \Lambda_r \Delta(\tau_r) \Delta[\tau(x_r|x')] p_{er}(x', t') \quad (14c)$$

$$p_q^r(x, t) = \Delta[\tau(x|x_q)] \oslash \Lambda_q \Delta(\tau_q) \Delta[\tau(x_q|x')] p_{eq}(x', t') \quad (14d)$$

The formalism for a single system is now summarized:

$$p(x, t) = h(x, x', t|t') p_e(x', t'),$$

$$p_e(x', t') = p_{er}(x', t') + p_{eq}(x', t');$$

$$h(x, x', t|t') = h_d(x, x', t|t') + h_{rev}(x, x', t|t'),$$

$$h_d(x, x', t|t') = \Delta[\tau(x|x')] = \Delta(|x - x'|/u^r) U(x' - x)$$

$$+ \Delta(|x - x'|/u^q) U(x' - x),$$

$$(\text{for } x_r > x_q)$$

$$\begin{aligned}
h_{rev} &= h_r^r + h_q^r + h_r^q + h_q^q = \sum_{\alpha, \beta}^{r, q} h_{\alpha}^{\beta} , \\
h_{\beta}^{\beta}(x, x', t|t') &= \Delta [\tau(x|x_{\alpha})] \Lambda_{\alpha} \Delta(\tau_{\alpha}) \Delta [\tau(x_{\alpha}|x_{\beta})] \\
&\quad \mathcal{D} \Lambda_{\beta} \Delta(\tau_{\beta}) \Delta [\tau(x_{\beta}|x')] \\
h_{\alpha}^{\beta}(x, x', t|t') &= \Delta [\tau(x|x_{\alpha})] \mathcal{D} \Lambda_{\alpha} \Delta(\tau_{\alpha}) \Delta \Delta [\tau(x_{\alpha}|x')] ; \\
\mathcal{D} &= \sum_{n=0}^{\infty} [\Lambda_r \Lambda_q \Delta(\tau_{sys})]^n , \\
\tau_{sys} &= \tau_{\alpha} + L/u^{\alpha} + \tau_{\beta} + L/u^{\beta} , \\
L &= |x_r - x_q| .
\end{aligned} \tag{15}$$

## RESPONSE OF SINGLE SYSTEM: AN EXAMPLE

The formalism presented in the previous section is now illustrated by defining the parameters for a representative (and simple) system and performing two sets of calculations of the temporal response for two examples of transient excitations.<sup>2</sup> One excitation is a pulse which is short compared with the propagation time associated with the system length,  $\tau_0$ , and the other is a tone burst which is relatively long compared with  $\tau_0$ .

The system parameters which are used for both examples are:

---

<sup>2</sup>The calculations were performed on a desk-top computer using the symbolic programming language "Mathematica". A symbolic processor of this nature offers considerable advantage in using the operator formalism defined here.

$$\begin{aligned}
u^r &= u^q = u, & (\text{the system is isotropic}), \\
x_q &= 0, \\
L/u &= \tau_0, & (\text{the temporal "length" of the system}), \\
\eta_r &= \eta_q = \eta = 0.01, & (\text{loss factors}), \\
\Lambda_r &= 0.8, \Lambda_q = 0.9, & (\text{reflection coefficients}), \\
\tau_r &= \tau_q = 0.0, & (\text{no delays during reflection}) \\
x'/L &= 0.33, \\
x/L &= 0.71.
\end{aligned} \tag{16}$$

The drive used for the first example is<sup>3</sup>

$$\begin{aligned}
p_e(x', t') &= \exp(-20|\bar{t}'|) \cos(50\bar{t}') \delta(x - x'), \\
\bar{t}' &= t'/\tau_0,
\end{aligned} \tag{17}$$

and is shown plotted as a function of time in Figure (2a). The time scale in this and all subsequent figures in this section is normalized by  $\tau_0$ . The direct temporal response at the point  $x$  (c.f. equation (11).) is shown in Figure (2b). The time scale is condensed compared with that of Figure (2a) and the origin ( $t = 0$ ) represents the time  $t = t' = 0$  when the center of the pulse is applied. It can be observed that the pulse at  $x$  is attenuated compared with the drive at  $x'$ . One of the reverberant response components,  $p_r^i$  (c.f. equation (14).) is shown in Figure (2c) on a time scale which is still further condensed to show several reverberations. The decrease in amplitude with reflection and propagation loss and the strict periodicity in the reverberant response are clearly evident. Finally, the combined response of the direct and all four reverberant components is shown in Figure (2d). During the first few reverberation times, the various individual returns are clearly discernable and one could identify an individual peak as being a member of a particular family defined by equation (14). After several reverberation times have elapsed, the system settles into a modal response. This is more clearly illustrated in Figure (2e) where the total response is

---

<sup>3</sup>The formalism does not require that the drive be a "good" function of time; only that it be represented functionally in such a way that the impulse response operator,  $h$ , sees the variable  $t$  on which to operate, and the result be calculable. The drive could, for example, be a functional representation of an experimentally measured drive.

again shown on a time scale which has been still further extended and the amplitude of the response has been plotted on a logarithmic scale to better accommodate the extreme attenuation.

For the second example, a relatively long "tone burst" is used as an external drive, and all the system parameters have been kept the same, including the location of the drive and observation points. The drive used is

$$p_e(x', t') = \{ \exp(-|\bar{t}'|) \cos(20 \bar{t}') / (|\bar{t}'| + 1) \} \delta(x - x') , \quad (18)$$

and is shown in Figure (3a).

In contrast to the drive of the first example, which was short and impulsive in nature, the drive considered here is long, on the order of a reverberation time for the system. The direct component is shown in Figure (3b) and the  $p_r^f$  component of the reverberant response is shown in Figure (3c). It can be noted that with the extended pulse, the system very quickly settles into its modal response. When all the response components are included, as in Figure (3d), the system settles in to its modal response even quicker. The final figure in this series, Figure (3e), shows the long time behavior of the system. Comparison of this figure with Figure (2e) shows that at long times after the impulse, the system response becomes modal and independent of the shape of the impulse.

The temporal response of a system which is not stationary in time is chosen as the last of the single system examples. Figure (4a) shows the length vs. time behavior of a hypothetical system which varies sinusoidally from initial length  $L = L_0$  to  $L_0(1 \pm 0.9)$ . The temporal response of this system to a "short" pulse, equation (17), is shown in Figure (4b). In order for both the drive and observation points to be within the system at its shortest length, they were placed at  $x'/L_0 = 0.03$  and  $x/L_0 = 0.07$ , respectively. Otherwise, the system specifications are given by equation (16). One could pick any parameter or combination of parameters in equation (16) to vary

with time and the formalism does not require that their variations be specified analytically; i.e., they could be imperically derived.

## MULTIPLE SYSTEMS

The formalism for a single system, presented in Section 3, will now be extended to multiple connected systems. A set of systems is depicted in Figure (5). For purposes of developing the formalism, it is required that the individual systems be spatially one-dimensional and each have a propagation velocity which is not a function of the spatial coordinate. Now, however, each system is in an environment consisting of other systems with which it interacts. These interactions are only through the junctions connecting the interacting systems. In the spatial formalism [1,2], the set of junctions is formulated as the (square) matrices  $\underline{\Lambda}_r$  and  $\underline{\Lambda}_q$ . The cross terms in the matrices, e.g.  $\Lambda_{rij}$ ,  $i \neq j$ , describe the interaction between different systems, e.g. from the (j)th to the (i)th at the r(th) junction; and the diagonal terms, e.g.  $\Lambda_{rjj}$ , describe the "self-interaction." The former quantities are generally termed the transmission coefficients whereas the latter are the reflection coefficients, and for conservative junctions,

$$\begin{aligned} \sum_i \Lambda_{\alpha ji} &= 1 \\ \alpha &= r \text{ or } q \end{aligned} \quad (19)$$

In the current formalism, the junctions may also be characterized with delays upon reflection and/or transmission. Accordingly, the junction delay matrix is defined.

$$\underline{\tau}_\alpha = (\Lambda_{\alpha ji} \tau_{\alpha ji}) \quad , \quad (20)$$

where  $\tau_{\alpha ji}$  is the delay experienced by the pulse when passing from the i(th) to the j(th) system. Thus, the off-diagonal elements of  $\underline{\tau}_\alpha$  give the attenuation and delay during transmission between

systems and the diagonal elements give the reflection coefficients and delays. The delay operator is defined to distribute over the elements of a matrix, thus, e.g.,

$$\Delta(\underline{\tau}_\alpha) = (\Lambda_{\alpha ji} \Delta(\tau_{\alpha ji})) \quad . \quad (21)$$

Each system, e.g. the (j)th, is terminated by end points  $x_{rj}$  and  $x_{qj}$ , and therefore is of length  $L_j = |x_{rj} - x_{qj}|$ . These sets of quantities are formulated in the vectors  $\underline{x}_r$ ,  $\underline{x}_q$  and  $\underline{L}$  with  $\underline{L} = |\underline{x}_r - \underline{x}_q|$ . The set of pulse propagation velocities associated with the systems are cast into the diagonal matrices  $\underline{u}^r$  and  $\underline{u}^q$ . The drive and observation points are also cast into the vectors  $\underline{x}'$  and  $\underline{x}$  and it should be noted that if a complex is driven (or observed) in only one system, then the drive vector  $\underline{x}'$  (or observation vector  $\underline{x}$ ) has only one non-zero element. The total system delays are cast into diagonal matrices whose elements, e.g. the (j)th, are the delay operators associated with a traverse of the length of the (j)th system, i.e.

$$\Delta[\underline{\tau}(\underline{x}_\beta | \underline{x}_\alpha)] = (\Delta[\tau(x_{\beta i} | x_{\alpha j})]) \delta_{ij} \quad . \quad (22)$$

The reverberation operator corresponding to equation (13b) can now be written,<sup>4</sup>

$$\underline{\mathcal{D}}_\alpha = \sum_{n=0}^{\infty} [\Delta[\underline{\tau}(\underline{x}_\alpha | \underline{x}_\beta)] \Delta(\underline{\tau}_\beta) \Delta[\underline{\tau}(\underline{x}_\beta | \underline{x}_\alpha)] \Delta(\underline{\tau}_\alpha)]^n \quad . \quad (23)$$

This is the generalization of equation (13b) and analogous to the factor  $\underline{D}$  in references 1 and 2.

With these definitions, the formalism for the multiple connected complex can be stated. The operator  $\underline{h}$  is sought for which

$$\underline{p}(\underline{x}, t) = \underline{h}(\underline{x}, \underline{x}', t | t') \underline{p}_e(\underline{x}', t') \quad , \quad (24)$$

---

<sup>4</sup>The matrix reverberation operator, equation (23), must carry an  $r$  or  $q$  subscript since, in general, the matrix operators involved do not commute.

where the  $h_{ji}$  element describes the response in the (j)th system due to a drive in the (i)th system. The drive is again separated into components emanating in the two directions, i.e.

$$\underline{p}_e(\underline{x}', t') = \sum_{\alpha=r,q} \underline{p}_{e\alpha}(\underline{x}', t') \quad (25)$$

Again, the separation into direct and reverberant components,

$$\underline{h}(\underline{x}, \underline{x}', t | t') = \underline{h}_d(\underline{x}, \underline{x}', t | t') + \underline{h}_{rev}(\underline{x}, \underline{x}', t | t') \quad (26)$$

with the direct impulse response operator,

$$\begin{aligned} \underline{h}_d(\underline{x}, \underline{x}', t | t')_{ji} &= \Delta[\underline{\tau}_{ji}(\underline{x}_j | \underline{x}'_i)] = \Delta[(\underline{u}^r)^{-1} | \underline{x}_j - \underline{x}'_i |] \underline{U}(\underline{x}_j - \underline{x}'_i) \underline{\delta}_{ji} \\ &+ \Delta[(\underline{u}^q)^{-1} | \underline{x}_j - \underline{x}'_i |] \underline{U}(\underline{x}'_i - \underline{x}_j) \underline{\delta}_{ji} \quad , \\ &(\text{for } x_{rj} > x_{qj}) \quad . \end{aligned} \quad (27)$$

The four reverberant impulse response operators are:

$$\begin{aligned} \underline{h}_{\alpha}^b(\underline{x}, \underline{x}', t | t') &= \Delta[\underline{\tau}(\underline{x} | \underline{x}_{\alpha})] \Delta(\underline{\tau}_{\alpha}) \underline{\mathcal{D}}_{\alpha} \Delta[\underline{\tau}(\underline{x}_{\alpha} | \underline{x}')] \quad , \\ \underline{h}_{\alpha}^a(\underline{x}, \underline{x}', t | t') &= \\ &\Delta[\underline{\tau}(\underline{x} | \underline{x}_{\beta})] \Delta(\underline{\tau}_{\beta}) \underline{\mathcal{D}}_{\beta} \Delta[\underline{\tau}(\underline{x}_{\beta} | \underline{x}_{\alpha})] \Delta(\underline{\tau}_{\alpha}) \Delta[\underline{\tau}(\underline{x}_{\alpha} | \underline{x}')] \quad . \end{aligned} \quad (28)$$

## RESPONSE OF MULTIPLE SYSTEMS: AN EXAMPLE

A simple example of the formalism presented in the foregoing section is illustrated by calculating the temporal response of a complex consisting of two connected systems excited by an impulse. Two sets of calculations are again presented; one for the case when the impulsive excitation is a short pulse and one for which it is a relatively long tone burst. The format of presentation will follow the foregoing examples illustrating the single system.

The system parameters used for the examples are:

$$\begin{aligned}
 \underline{u}^r &= \underline{u}^q = \underline{u}, & (\text{the systems are isotropic}), \\
 \underline{x}_q &= \{0, 0\}, \\
 \underline{u}^{-1} \underline{L} &= (\tau_0, 1.4 \tau_0), & (\text{the temporal "lengths" of the systems}), \\
 \underline{\eta}_r &= \underline{\eta}_q = \underline{\eta} = \{0.005, 0.005\}, & (\text{loss factors}), \\
 \underline{\Lambda}_r &= \begin{pmatrix} 0.55 & 0.4 \\ 0.45 & 0.6 \end{pmatrix}, \\
 \underline{\Lambda}_q &= \begin{pmatrix} 0.45 & 0.35 \\ 0.55 & 0.65 \end{pmatrix}, & (\text{junction matrices}), \\
 \underline{\tau}_r &= \underline{\tau}_q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & (\text{no delays upon reflection} \\
 & & \text{or transmission between systems}), \\
 \underline{x}' &= \{0.33 L_1, 0\}, & (\text{the complex is driven in system \#1}), \\
 \underline{x} &= \{0.75 L_1, 0\}, & (\text{the response is assessed in system \#1}).
 \end{aligned} \tag{29}$$

The drives for the two examples are chosen to be the same as for the single system examples, i.e. a short pulse and a long pulse given by equations (14) and (15) respectively, and shown in Figures (2a) and (3a) respectively. Since both the observation and the drive points are chosen to be in the same system and are separated by the same amount in both sets of examples, the direct temporal responses for the two system examples are the same as in Figures (2b) and (3b) respectively and these figures are not repeated. The reverberant response components *are* different from the single system examples. The  $p_f^r$  reverberant component for the short pulse, equation (17) is shown in Figure (6a). All components of the response, the direct plus the four reverberant for the short pulse excitation, are shown in Figure (6b). This total response is shown again in Figure (6c) with the amplitude and time scales extended to 20 times the pulse transit time for the system which contains the drive.

The same figures for the case using the long pulse drive, equation (18), are given in Figure 7; i.e. Figure (7a) shows the  $p_f^r$  reverberant component and Figures (7b) and (7c) show the total response in the formats of Figures (6b) and (6c) respectively.

The temporal response as a function of two variables,  $t$  and  $x$ , is shown in Figure (8). Figure (8a) shows the spatial and temporal evolution of the response in the driven system to a short pulse, equation (17). The drive is applied at  $t' = 0$  at  $x'_1/u_{11} = 0.33 \tau_0$  (in system number 1) and is shown propagating away from the drive point and interacting with the second system at the junctions. The parameters for the complex are given in equation (29) with the exception that the observation point is not fixed but rather used as a variable. Careful observation of Figure (8a) reveals response contributions which have traversed the second system and returned to the first system; they appear as pulses entering from the junctions. The response in the second, undriven, system is shown in Figure (8b).

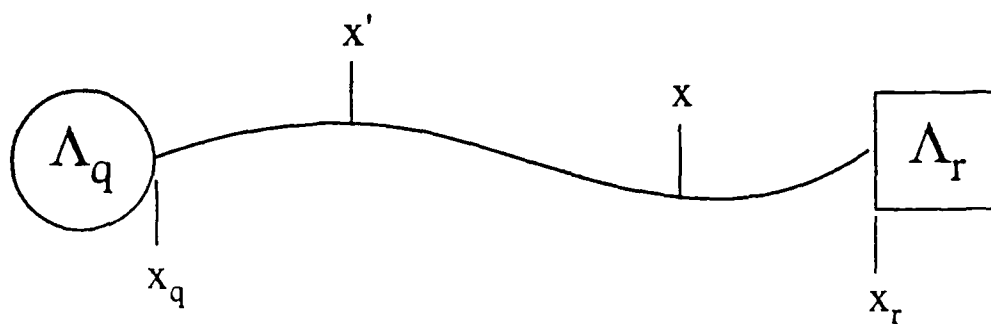


Fig. 1. Schematic of a single one-dimensional dynamic system with terminal coordinates  $x_r$  and  $x_q$  and reflection coefficients  $\Lambda_r$  and  $\Lambda_q$  at the end points. The drive point and observation point are given by  $x'$  and  $x$  respectively.

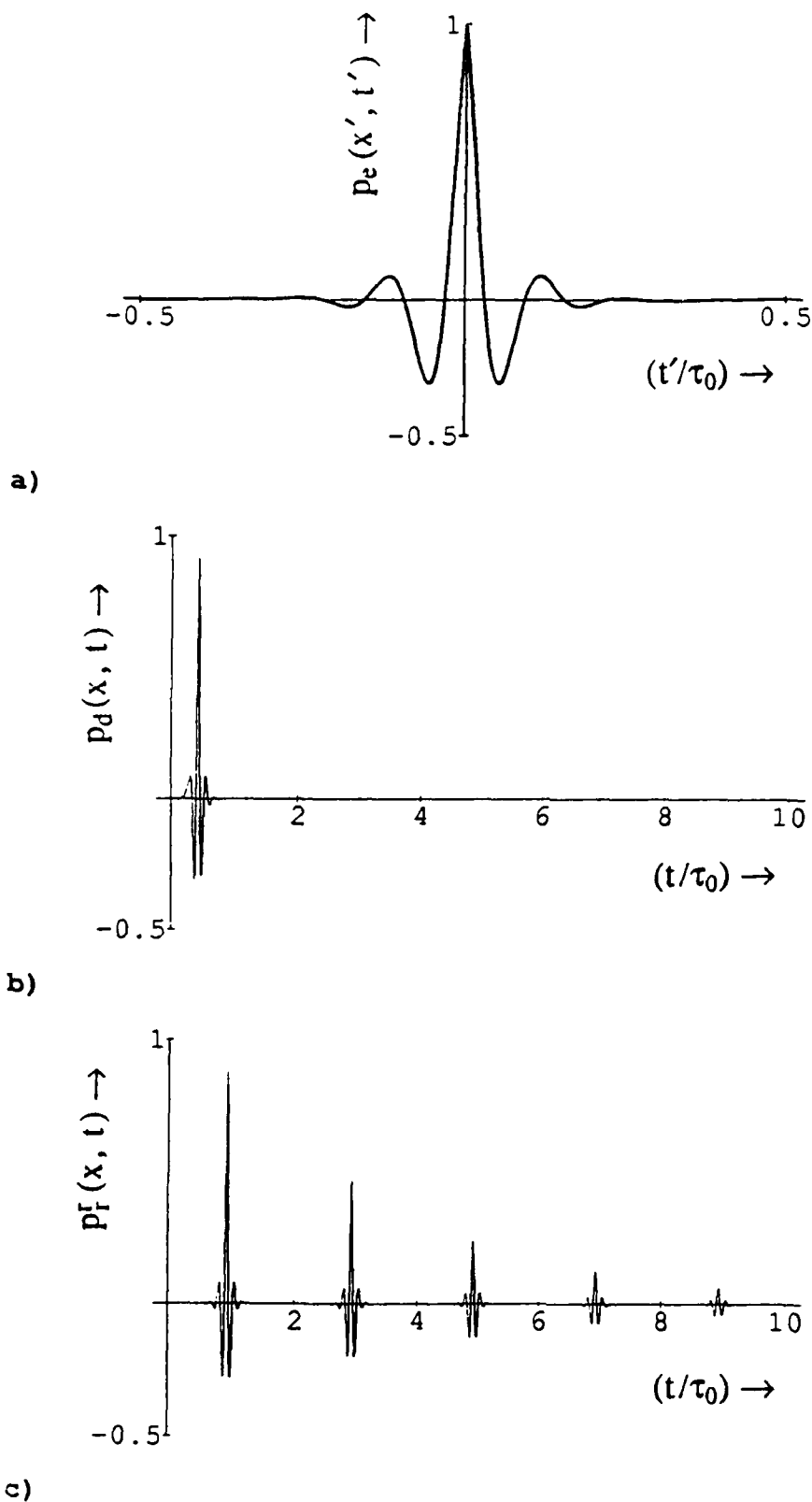
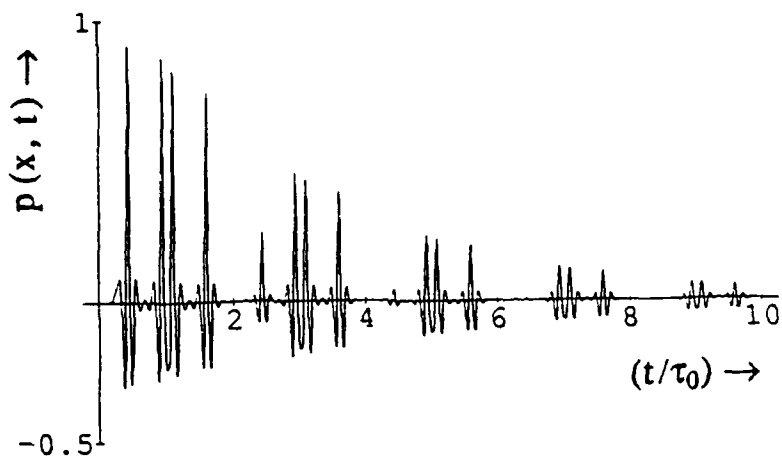


Fig. 2. The single system as defined by equation (16) responding to a "short", unit amplitude pulse defined by equation (17). The normalizing time,  $\tau_0$ , is the one-way transit time for a pulse in the system.

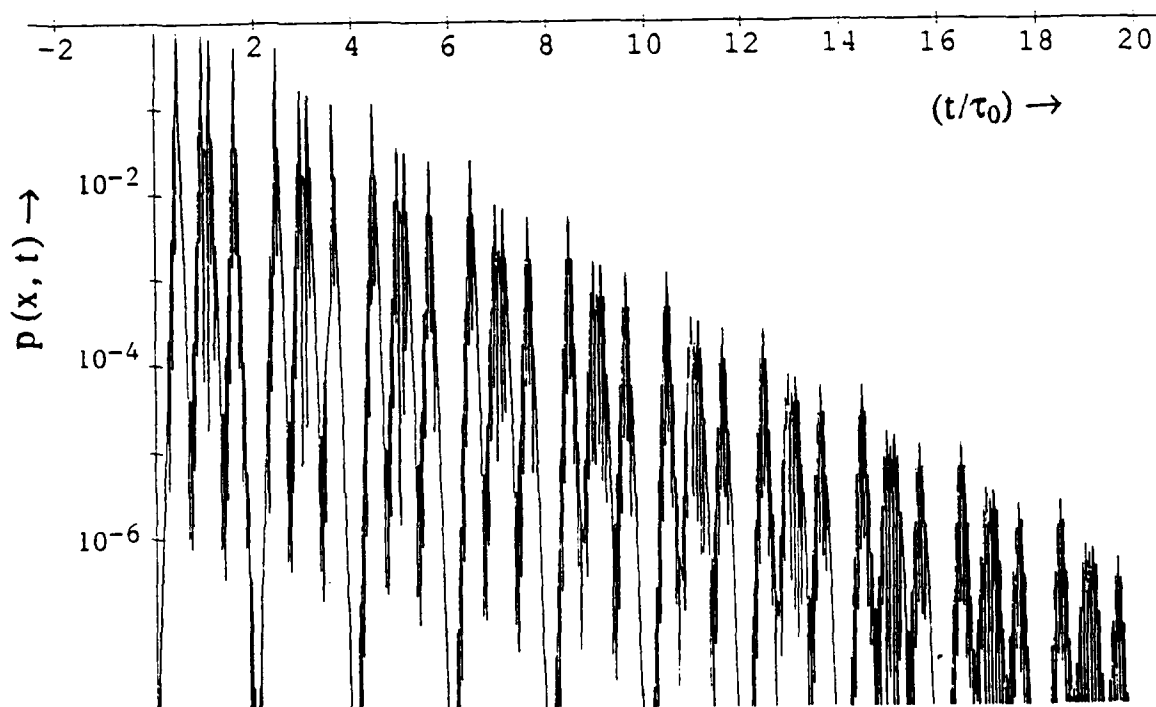
(a) The drive pulse applied at  $t = t' = 0$  and defined by equation (17).

(b) The direct response.

(c) The  $p_r^f$  component of the reverberant response; i.e. the response which results from the drive component emanating from the point  $x'$  towards the  $r$  junction (c.f. Fig. 1) and which arrives at the point  $x$  traveling towards the  $r$  junction.



d)



e)

Fig. 2. (Continued)

(d) The total response; i.e., direct plus all four reverberant components.

(e) The same as (d) except that the amplitude and the time scales are extended.

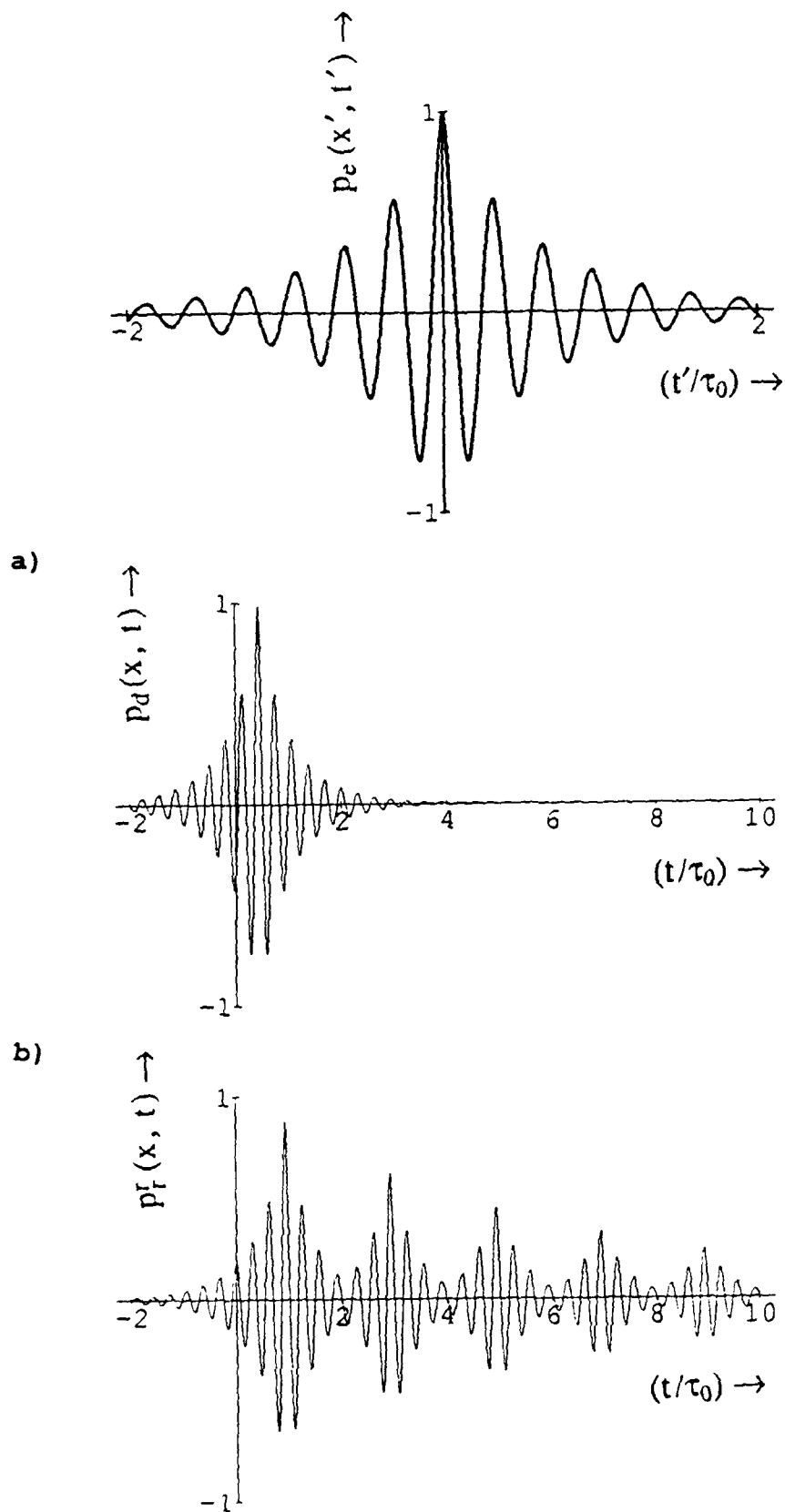
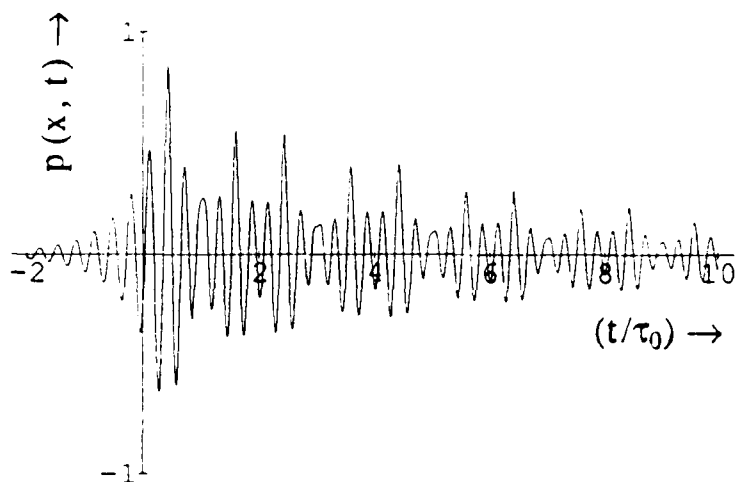
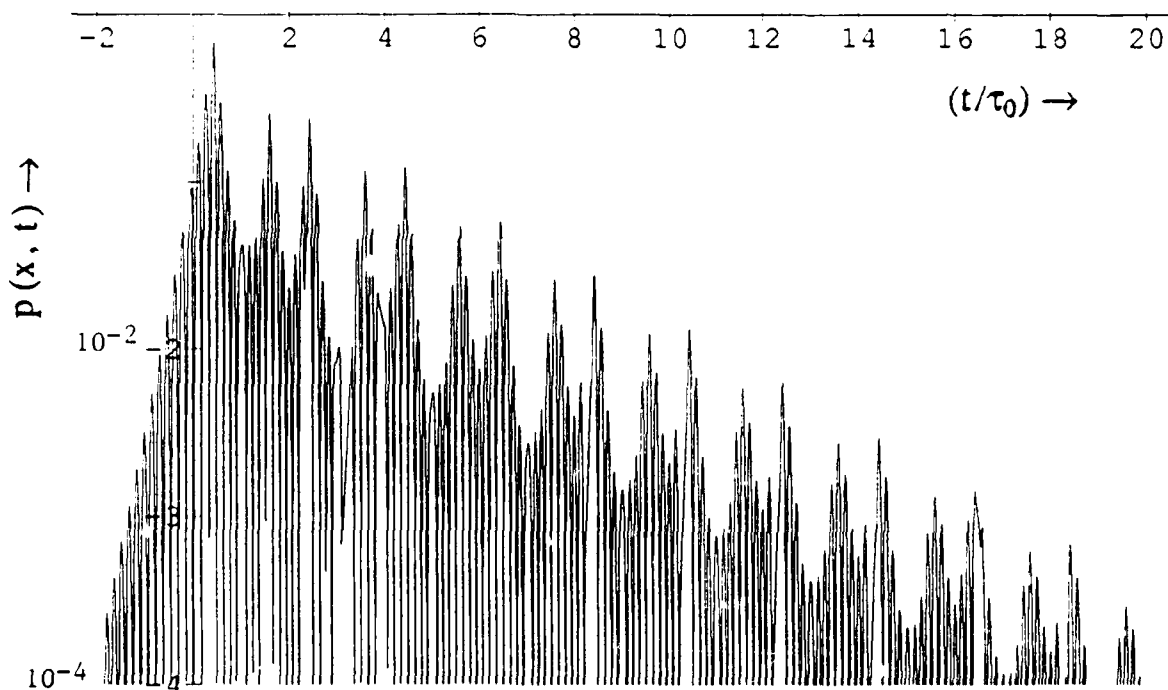


Fig. 3. The same as Fig. 2 except that the drive is a "long" unit amplitude pulse defined by equation (18).

- (a) The drive pulse.
- (b) The direct response.
- (c) The  $p_r^f$  component of the reverberant response.



d)

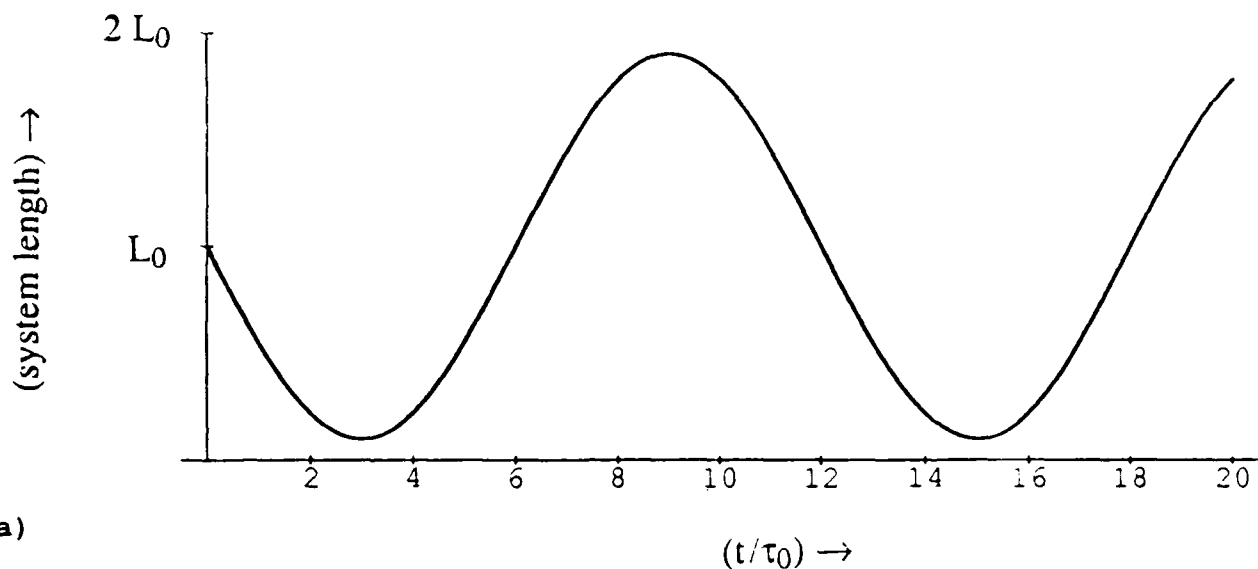


e)

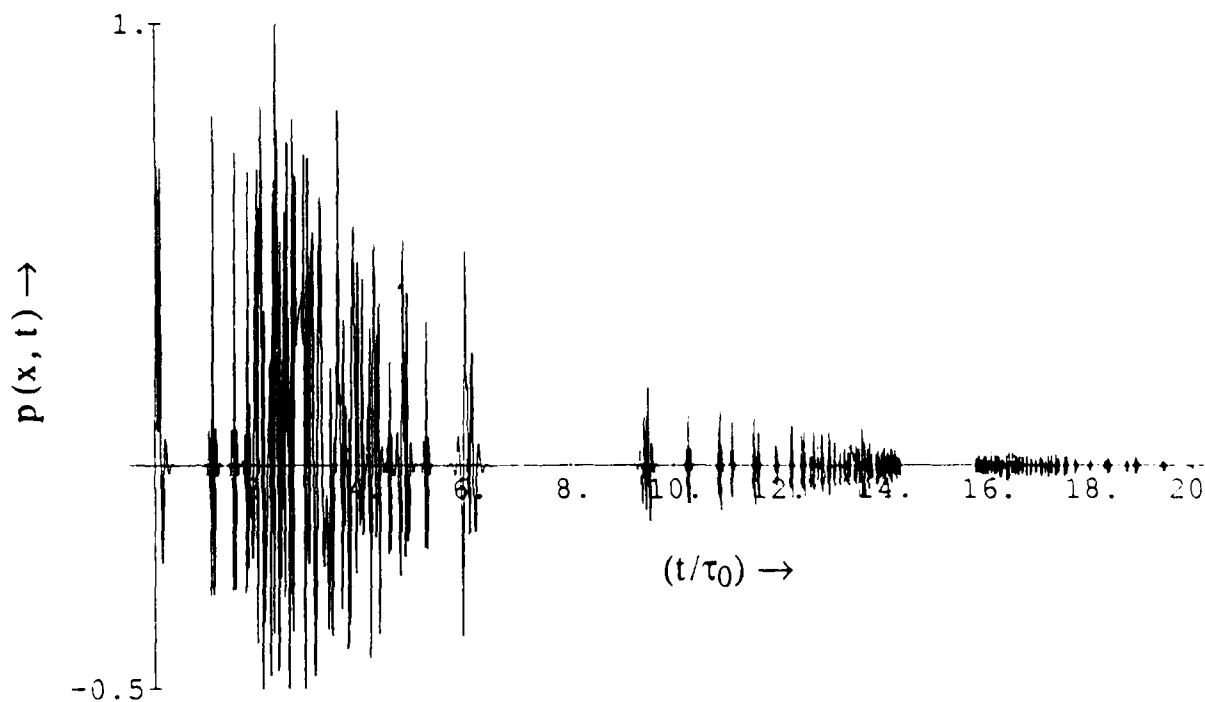
Fig. 3. (Continued)

(d) The total response.

(e) The total response in the format of Fig. 2e.



a)



b)

Fig. 4. The single system with length varying with time.

(a) The length of the system as a function of time.

(b) The total response, i.e., direct plus all four reverberant components for the system. The system is driven at  $(x'/L_0) = 0.03$  and the response is assessed at  $(x/L_0) = 0.07$ . The reflection coefficients are  $\Lambda_r = \Lambda_q = 0.95$ . The other parameters are specified in equation (16).

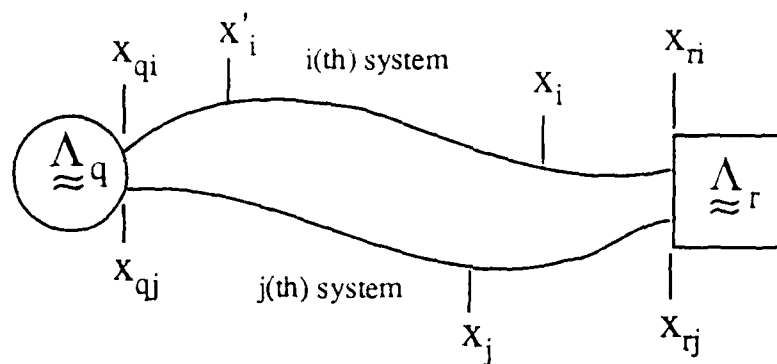


Fig. 5. Schematic of a complex of one-dimensional systems. There may be any number of systems in the complex.

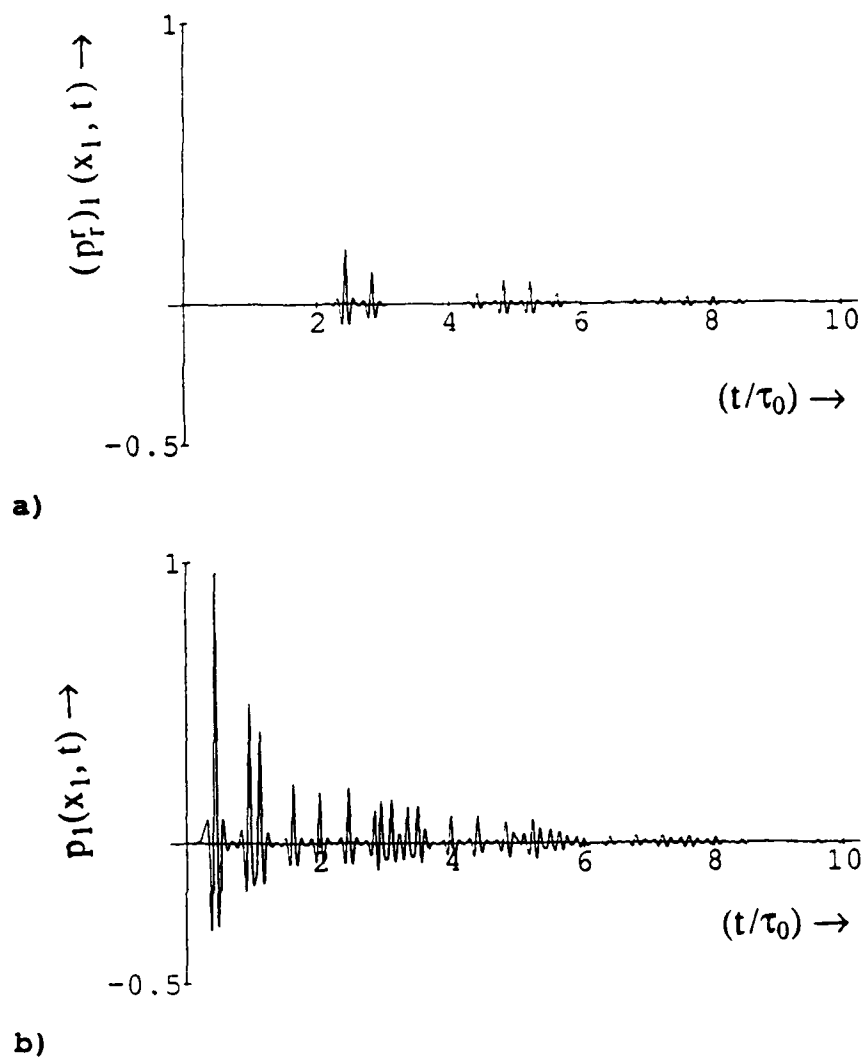
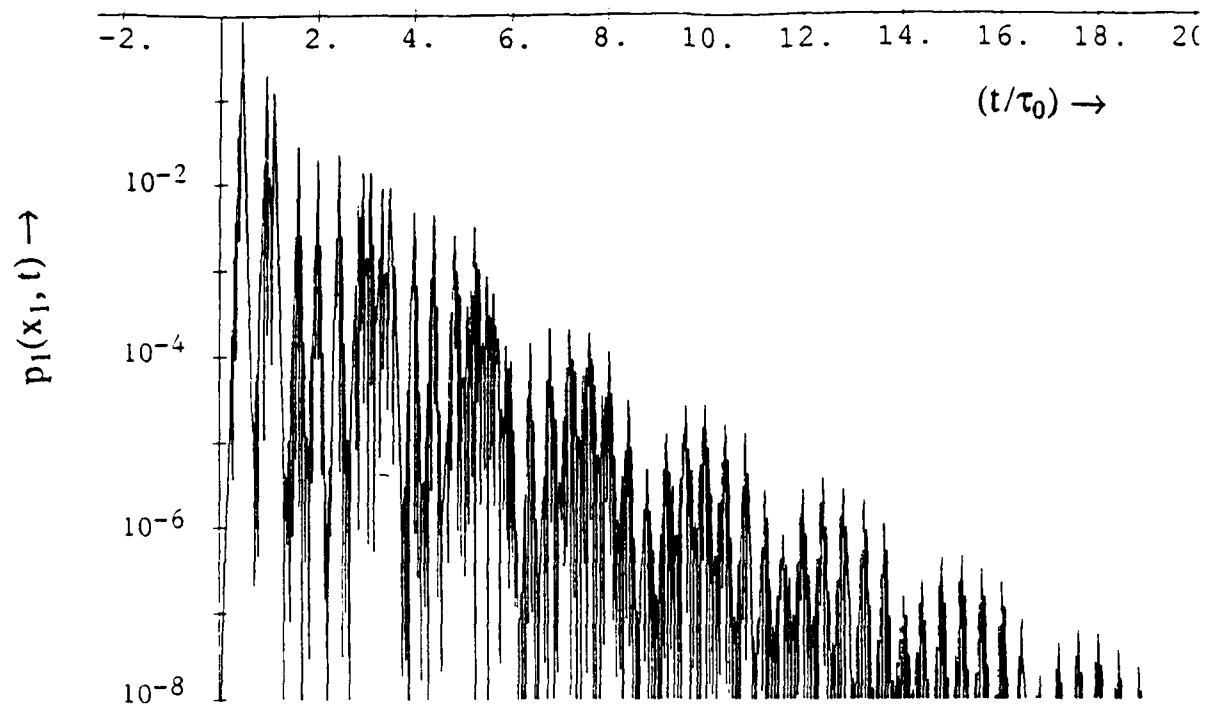


Fig. 6. The complex as defined by equation (29) responding to a "short" unit amplitude pulse defined by equation (17), and shown in Fig. 2a., applied in system 1. The normalizing time,  $\tau_0$ , is the one-way transit time for a pulse in system number 1.

(a) The  $(p_r')_1$  component of the response; i.e. the response in system number 1 which results from the drive component emanating towards the r junction, the drive being applied at the point  $x'_1$  in system number 1, and which arrives at the point  $x'_1$  traveling towards the r junction.

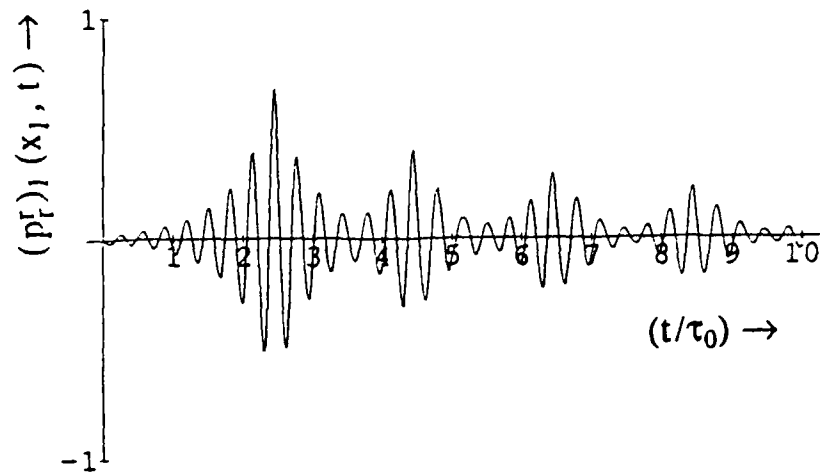
(b) The total response; i.e. the direct plus all four reverberant components.



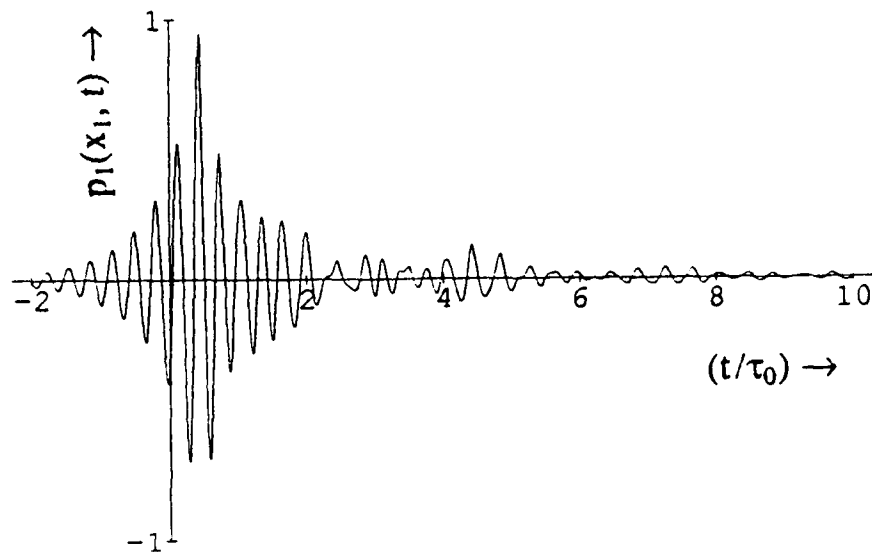
c)

Fig. 6. (Continued)

(c) The same as (b) except in the format of Fig. 2e.

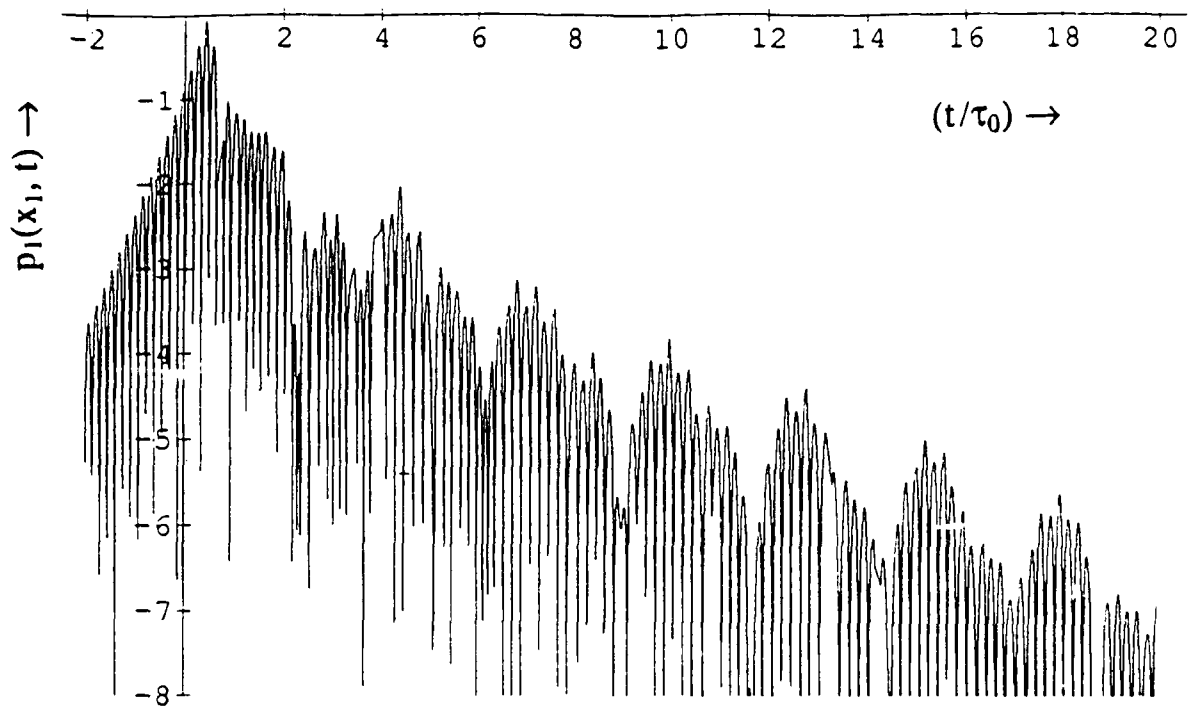


a)



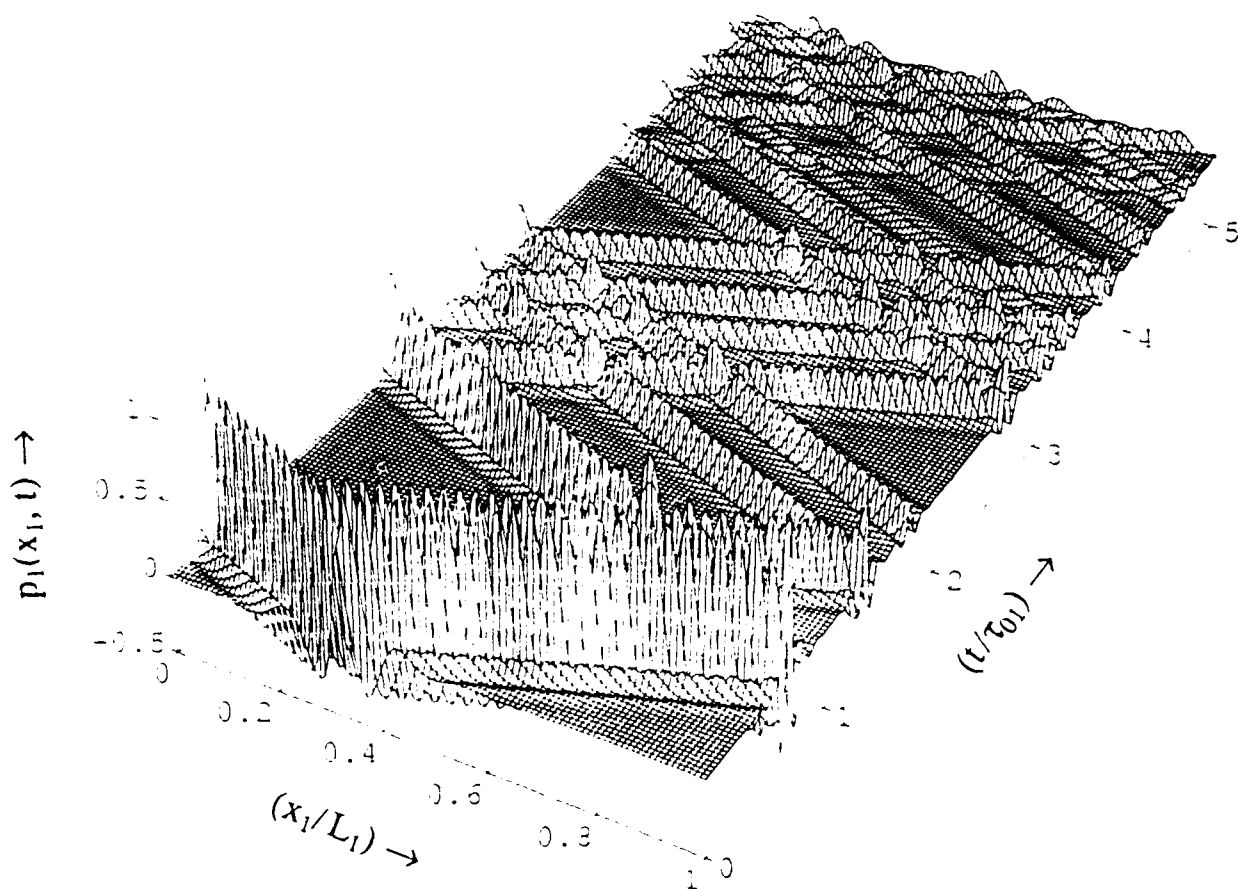
b)

Fig. 7. The same as Fig. 5 except that the drive is a "long" unit amplitude pulse defined by equation (18) and shown in Fig. 3a.  
 (a) The  $(p_r^r)_1$  component of the response.  
 (b) The total response.



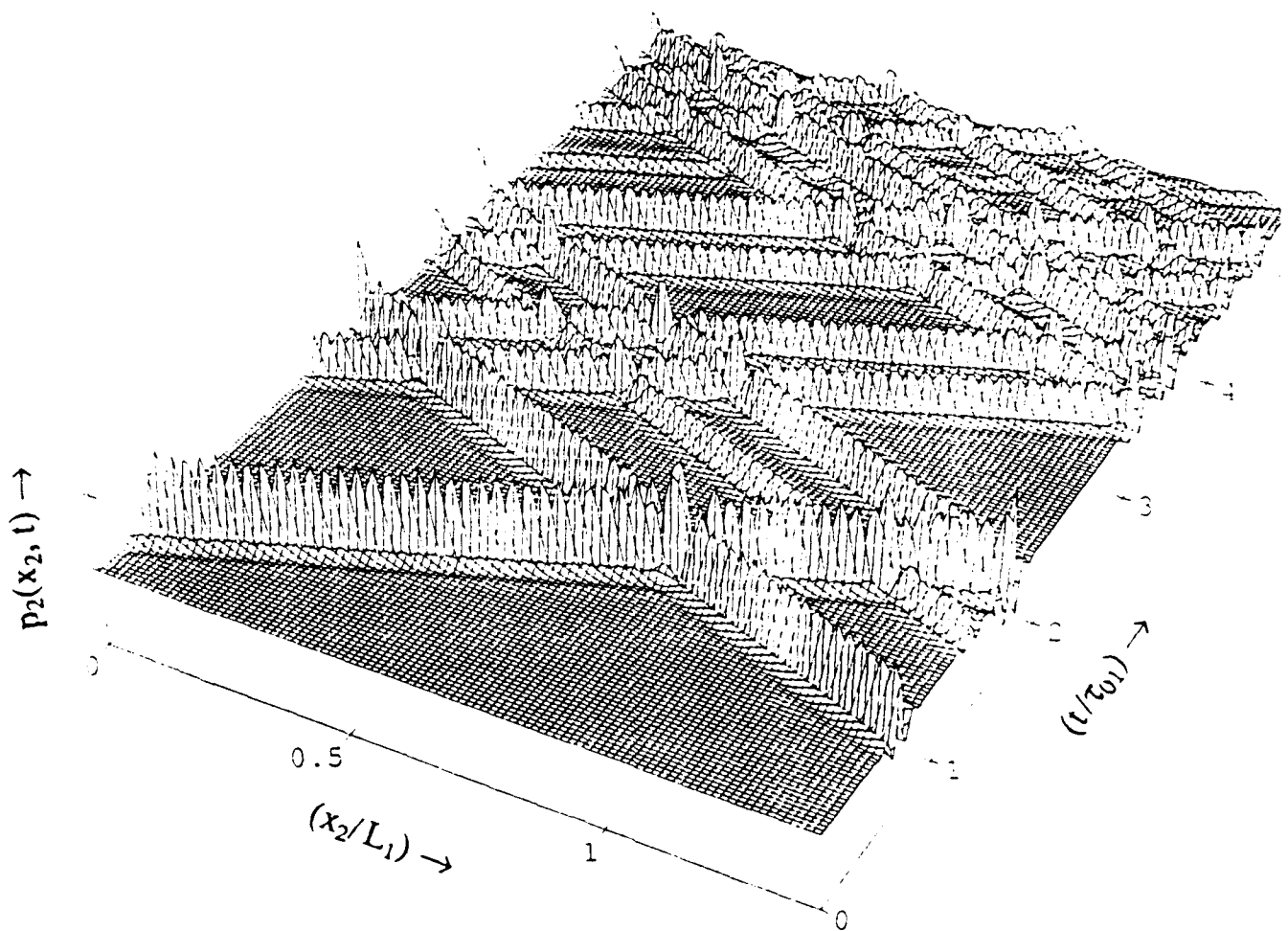
c)

Fig. 7. (Continued)  
(c) The total response in the format of Fig. 5c.



a)

Fig. 8. The response of a two system complex as a function of the normalized spatial coordinate in the driven system and the normalized time.  
(a) The system containing the drive.



b)

Fig. 8. (Continued)  
 (b) The system which does not contain the drive.

## REFERENCES

1. L. J. Maga and G. Maidanik, 1983, *Journal of Sound and Vibration*, 86, 473-488. Response of multiple coupled dynamic systems.
2. G. Maidanik and L. J. Maga, 1986, *Journal of Sound and Vibration*, 111, 361-375. Response of coupled one-dimensional dynamic systems.
3. G. Maidanik and J. Dickey, 1990,. Impulse response operators for structural complexes. In preparation
4. G. Maidanik and J. Dickey, 1990, *Journal of Sound and Vibration*, 136 (1), 105-119. Resonances of narrow one-dimensional cavities.

## INITIAL DISTRIBUTION

Copies		Copies		
2	ONT (Remmers)	1	1908	(McKeon)
3	ONR	1	1926	(Keech)
	(Hansen)	1	195	
	(Reischmann)	1	196	
	(Abrambson)	1	1961	
3	NAVSEA	1	1962	
	2 SEA 55N (Biencardi)	1	1965	
	1 SEA 92R			
1	DARPA	1	27	
12	DTIC	20	2704	
6	USNA	1	2712	(Brown)
	3 Mathematics Department	1	274	
	(Davis, McCoy, D'Archangelo)	1	2741	
	1 Physics Department	1	2742	
	(Ertel)	1	2743	
	1 Electrical Engineering	1	2744	
	(Sarkady)	1	2749	
	1 Aerospace Engineering	1	2753	(Whitesel)
	(Rogers)			
		1	2840	(Schumacher)
CENTER DISTRIBUTION				
1	01A	1	342.1	TIC(C)
1	0113	1	342.2	TIC(A)
1	17	2	3431	
1	172	10	3432	Reports Control
1	18			
1	19			
1	1905.1 (Blake)			